An Annalus Approach To Fuzzy Subgroups

by

Khalid Abdul-Aziz Holayan Al-Shammari

A Thesis Presented to the

FACULTY OF THE COLLEGE OF GRADUATE STUDIES

KING FAHD UNIVERSITY OF PETROLEUM & MINERALS

DHAHRAN, SAUDI ARABIA

In Partial Fulfillment of the
Requirements for the Degree of

MASTER OF SCIENCE

In

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INFORMATION TO USERS

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THESIS ABSTRACT

FULL NAME OF STUDENT Khalid Abdul Aziz Holayan Al-Shammari

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In this thesis we introduce the annuli of a fuzzy subgroup and use it as a tool to give alternate proofs of some properties of fuzzy subgroups and to straighten out some known results on fuzzy normal and fuzzy abelian groups and obtain some new results about them. It is also used to describe the left and right fuzzy cosets determined by a fuzzy subgroup.

We obtain some commutative diagrams which relate to homomorphisms, fuzzy subgroups and fuzzy quotient groups. Finally, we straighten out some chains of fuzzy normal subgroups which lead to the concept of fuzzy solvable groups.

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خلاصة الرسالة

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في هذه الرسالة قدّمنا مفهوم حلقات الزمر الفازية وإستخدمناها في إثبات الخواص والنتائج المتعلقة بالزمر الفازية الناظمية والزمر الفازية الإبدالية ، أيضاً إستخدمناها في إعطاء وصف كامل للمجموعات الفازية المصاحبة اليسرى واليمني .

كما توصلنا إلى مخططات إبدالية والتي تربط الهومومورفزم وزمر القسمة الفازية بالزمر الفازية . وأخيراً قمنا بتقديم تعريف معدّل للزمر الفازية القابلة للحل وما يترتب عليها من نتائج .

درجة الماجستير
في العلوم الرياضية
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INTRODUCTION

The close of interconnection between the structure of a group G and the level subgroups associated with any fuzzy subgroup μ of G, prompted many researchers in this area to characterize various types of fuzzy subgroups and/or to obtain many of their useful properties by using level subgroups as the main tool, as can be seen in [8], [10], [17], and [22].

In this thesis, we study further the concepts of fuzzy normal subgroups, fuzzy cosets, and fuzzy quotient. During our investigation, we have straightened out some incorrect results about these concepts. These concepts are generalizations of the important concepts of normal subgroups and quotient groups in the classical group theory, and consequently group homomorphisms must be involved in one way or another.

This thesis is composed mainly of five chapters. In chapter zero, we give a brief account of the properties of fuzzy subsets needed throughout the thesis. In chapter one, we introduce the notion of the annuli of a fuzzy subgroup which is derived from the notion of level subgroups, then we use it to give alternate proofs of the preliminary results on fuzzy subgroups needed for our work. It will be also used as a tool throughout the thesis. Chapter two is divided into two parts. In the first part, we study the various equivalent statements on the concept of fuzzy normal subgroups using an annulus approach. Then we show, by means of a counter-example (Example 2.2.1), that one of these statements is incorrect. We then give a correct version of it in Proposition 2.2.3. We also study the homomorphic image and inverse image of a fuzzy normal subgroup. In the second part of chapter 2, we use Lemma 2.3.1, Examples 2.3.1, 2.3.2 and Proposition 2.3.2 to straighten out some wrong assertions on fuzzy abelian groups. Then, we introduce the notion of the centralizer of a fuzzy subgroup and show that it is a normal subgroup. In the first part of chapter 3, the annulus approach is used to study the various properties of right and left cosets determined by a fuzzy subgroup and to obtain the structure of an annulus of a left or right coset. In the second part of chapter 3, we obtain a necessary and sufficient condition for an annulus of a fuzzy coset to be a subgroup. In chapter 4, we study the notions of fuzzy quotient groups and the quotient of fuzzy subgroups. We first follow the work of Mukherjee and Bhattacharya [7] to construct the group G/μ of fuzzy cosets determined by a given fuzzy normal subgroup μ of a group G and show that G/μ is a homomorphic image of G. Then we construct two commutative diagrams (Propositions 4.1.2 and Theorem 4.1.1) that describe the structure of the fuzzy quotient group $\overline{\mu}$ determined by μ by means of G, G/μ, and some isomorphism and canonical homomorphism. We conclude chapter 4 by discussing the material on fuzzy solvable subgroups introduced by Bhattacharya and Mukherjee [7] and straighten out the chain of fuzzy normal subgroups which they introduced to define fuzzy solvable groups. Then we give a necessary and sufficient condition for a fuzzy subgroup to be solvable.

It must be noted that we will deal mainly with finite groups and hence fuzzy subgroups of finite images. Most of the result in this thesis can be extended to fuzzy subgroups of finite images regardless the group is finite or infinite.

CHAPTER 0

Fuzzy Subsets

In this chapter. we give a brief account of the properties of fuzzy subsets which are going to be used throughout the thesis.

0.1 Fuzzy Subsets

Fuzzy set theory. introduced by Zadeh [28], is a generalization of abstract set theory. A subset A of a set X can be characterized by the characteristic function $\chi_A : X \to \{0, 1\}$. while a fuzzy subset $\tilde{A}$ of a set X is characterized by a membership function $\mu_{\tilde{A}} : X \to [0, 1]$. Usually, a fuzzy subset $\tilde{A}$ is identified with its membership function $\mu_{\tilde{A}}$ or simply μ. In this thesis, the unit interval $[0, 1]$ will be denoted by I. Thus. Zadeh's definition of a fuzzy subset can be formulated as follows:

Definition 0.1.1. Let X be a set. The map $\mu : X \to I$ is called a *fuzzy subset* of X. The set of all fuzzy subsets of X will be denoted by I^X. Notice that if A is a subset of A. then $\chi_A \in I^X$.

Operations on fuzzy subsets of a set X are given in the following definition:

Definition 0.1.2. Let μ and ν be fuzzy subsets of a set X. Then for all $x \in X$

we have

(i) Equality $\mu = \upsilon \Leftrightarrow \mu(x) = \nu(x)$.

(ii) Inclusion $\mu \subseteq \upsilon \Leftrightarrow \mu(x) \leq \nu(x)$.

(iii) Intersection $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$.

(iv) Union $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$.

(iv) Complement $\nu = \mu' \Leftrightarrow \nu(x) = 1 - \mu(x)$.

More generally, for a family of fuzzy subsets $\{\mu_i | i \in J\}$ of X, the union $\alpha = \bigcup_{i \in J} \mu_i$ and the intersection $\beta = \bigcap_{i \in J} \mu_i$ are defined by

$$\alpha(x) = \sup_{i \in J} \{\mu_i(x)\}, \qquad x \in X$$

and

$$\beta(x) = \inf_{i \in J} \{\mu_i(x)\}, \qquad x \in X.$$

The symbol ϕ will be also used to denote an empty fuzzy subset of X, i.e., $\phi(x) = 0$ for all $x \in X$.

It is worth mentioning that not all properties of ordinary subsets are true, in general, for fuzzy subsets; e.g., one can easily check that

(A) $\mu \cup \mu' \neq X, \quad \mu \cap \mu' \neq \phi$.

(B) If $\mu \cap \nu = \phi$, then $\mu \subset \mu'$, but $\mu \subset \nu'$ does not imply that $\mu \cap \nu = \phi$.

References [16], [30] may be consulted for more details on fuzzy subsets.

0.2 Fuzzy Subsets Induced by Functions

Consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mu"'] & Y \arrow[d, "\nu"] \\ I & I \end{tikzcd}$$

where f is a function from a set X to a set Y while μ and ν are fuzzy subsets of X and Y. respectively. Then we have the following definitions of fuzzy subsets induced by the function f:

Definition 0.2.1. The *image* $f(\mu)$ of μ under f is the fuzzy subset $f(\mu) : Y \rightarrow I$ defined by

$$f(\mu)(y) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi. \end{cases}$$

Definition 0.2.2. The *inverse image* $f^{-1}(\nu)$ of ν under f is the fuzzy subset $f^{-1}(\nu) : X \rightarrow I$ defined by $f^{-1}(\nu)(x) = \nu(f(x))$.

Definition 4 gives immediately that $f^{-1}(\nu) = \nu \circ f$ and the following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "f^{-1}(\nu)"'] & & Y \arrow[dl, "\nu"] \\ & I & \end{tikzcd}$$

The important properties of fuzzy subsets induced by functions are summarized in the following theorem. (see [18], [30]):

Theorem 0.2.1. *Consider the following diagram:*

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, shift right=2, dash, "\mu_1"'] \arrow[d, shift left=2, dash, "\mu_2"] & H \arrow[r, "g"] \arrow[d, shift right=2, dash, "\nu_1"'] \arrow[d, shift left=2, dash, "\nu_2"] & K \arrow[d, "\eta"] \\
I & I & I
\end{tikzcd}$$

Then

(A) $\nu_1 \subseteq \nu_2 \Rightarrow f^{-1}(\nu_1) \subseteq f^{-1}(\nu_2)$.

(B) $\mu_1 \subseteq \mu_2 \Rightarrow f(\mu_1) \subseteq f(\mu_2)$.

(C) $f\left(f^{-1}(\nu_1)\right) \subseteq \nu_1$.

(D) If f *is onto, then* $f\left(f^{-1}(\nu_1)\right) = \nu_1$.

(E) $\mu_1 \subseteq f^{-1}\left(f(\mu_1)\right)$.

(F) If f *is one-to-one, then* $f^{-1}\left(f(\mu_1)\right) = \mu_1$.

(G) $f^{-1}(\nu_1 \cap \nu_2) = f^{-1}(\nu_1) \cap f^{-1}(\nu_2)$.

(H) $f^{-1}(\nu_1 \cup \nu_2) = f^{-1}(\nu_1) \cup f^{-1}(\nu_2)$.

(I) $f(\mu_1 \cap \mu_2) \subseteq f(\mu_1) \cap f(\mu_2)$.

(J) If f *is one-to-one, then* $f(\mu_1 \cap \mu_2) = f(\mu_1) \cap f(\mu_2)$.

(K) $f(\mu_1 \cup \mu_2) = f(\mu_1) \cup f(\mu_2)$.

(L) $(g \circ f)(\mu_1) = g\,(f(\mu_1))$.

(M) $(g \circ f)^{-1}(\eta) = f^{-1}\,(g^{-1}(\eta))$.

0.3 Level Subsets

The notion of level subsets introduced by Zadeh [28] is as follows:

Definition 0.3.1. Let μ be a fuzzy subset of a set X and let $t \in I$. The subset μ_t of X given by $\mu_t = \{x \in X | \mu(x) \geq t\}$ is called a level subset of X. (It is also called a t-level subset of μ or a t-cut of μ.)

The notion of level subsets plays an important role in the application of the theory of fuzzy subsets in group theory as we will see in this thesis.

0.4 Applications of Fuzzy Subsets

Researchers have found many applications of fuzzy subsets in engineering, computer science. medical diagnosis. social behavioral science, etc. [11], [21], [26] and [29].

In the domain of mathematics. researchers in various disciplines have been trying to extend their ideas to the broader framework of fuzzy setting. As a result, a number of important mathematical structures have been fuzzified such as topological spaces. linear spaces, algebra, categories, groups, automata, graphs, probability, etc.. [1], [2], [3], [4], [5], [6], [8], [9], [12] and [19]. In this thesis, our interest is focused on fuzzy subgroups.

Chapter One

The Annuli of a Fuzzy Subgroup

In this chapter we introduce the notion of the annuli of a fuzzy subgroup and use it as a tool to prove some of the basic results which will be used in the rest of the thesis.

1.1 Fuzzy Subgroups: Definition and Examples.

In 1971, Rosenfeld [23] used the concept of fuzzy subsets to develop the theory of fuzzy subgroups. The definition of a fuzzy subgroup is as follows (see also [19]):

Definition 1.1.1. A fuzzy subset μ of a group G is a *fuzzy subgroup* of G if

(i) $\mu(xy) \geq \min(\mu(x), \mu(y))$ for all $x, y \in G$.

(ii) $\mu(x) = \mu(x^{-1})$ for all $x \in G$.

Thus, in a sense, the value of the fuzzy subgroup at a particular $x \in G$ represents the possibility that x will be found in a randomly selected subgroup.

Let e be the identity element of G, then the following proposition is an immediate consequence of Definition 1.1.1.

Proposition 1.1.1. *Let μ be a fuzzy subgroup of a group G. Then* *(i)* $\mu(x) \leq \mu(e)$ *for all* $x \in G$.

(ii) The subset $G_\mu = \{x \in G | \mu(x) = \mu(e)\}$ *is a subgroup of* G.

Proof. Let x be any element of G, then

$$\begin{aligned} \mu(x) &= \min(\mu(x), \mu(x)) = \min\left(\mu(x), \mu(x^{-1})\right) \\ &\leq \mu(xx^{-1}) = \mu(e) \end{aligned}$$

and (i) is proved. To prove (ii), we have $e \in G_\mu$, then $G_\mu \neq \phi$. Now let $x, y \in G_\mu$, then

$$\begin{aligned} \mu(xy^{-1}) &\geq \min\left(\mu(x), \mu(y^{-1})\right) = \min(\mu(x), \mu(y)) \\ &= \min(\mu(e), \mu(e)) = \mu(e). \end{aligned}$$

But from (i) $\mu(xy^{-1}) \leq \mu(e)$ for all $x, y \in G$. Therefore $\mu(xy^{-1}) = \mu(e)$, which means $xy^{-1} \in G_\mu$ and hence G_μ is a subgroup of G. This completes the proof of (ii).

The subgroup G_μ will be used throughout this thesis.

Now, we give two examples: one of a fuzzy subgroup of a finite image and the other of a fuzzy subgroup with infinite image.

Example 1.1.1. Let G be the Klein four-group, i.e., $G = \langle a, b | a^2 = b^2 = (ab)^2 = e \rangle$. One can easily check that the fuzzy subset μ of G defined by

$$\mu(e) = \frac{3}{4}, \;\; \mu(a) = \frac{1}{2} \;\; \text{and} \;\; \mu(b) = \mu(ab) = \frac{1}{3}$$

is a fuzzy subgroup of G as illustrated below.

μ	e	a	b	ab
e	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{3}$
a	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{1}{3}$
b	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{3}{4}$	$\frac{1}{2}$
ab	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{3}{4}$

Example 1.1.2 [3]. Let Z be the additive group of integers and let (2^n) denote, as usual, the set of multiples of 2^n, where n is a fixed positive integer. It is easy to check that the fuzzy subset μ of Z which is defined as follows

$$\mu(x) = \begin{cases} 0 & \text{for} \quad x \in Z - (2) \\ \frac{1}{2}\left(1 - \frac{1}{2^n}\right) & \text{for} \quad x \in (2^n) - (2^{n+1}), \quad n = 1, 2, 3, \ldots. \end{cases}$$

is a fuzzy subgroup of Z.

Proposition 1.1.2 [23]. *Let μ and ν be two fuzzy subgroups of a group G, then $\mu \cap \nu$ is a fuzzy subgroup of G.*

Proof. Let x and y be any two elements of G, then

$$\begin{aligned} (\mu \cap \nu)(xy) &= \min(\mu(xy), \nu(xy)) \\ &\geq \min(\min(\mu(x), \mu(y)), \min(\nu(x), \nu(y))) \\ &\geq \min(\min(\mu(x), \nu(x)), \min(\mu(y), \nu(y))) \\ &= \min((\mu \cap \nu)(x), (\mu \cap \nu)(y)). \end{aligned}$$

Also

$$\begin{aligned}(\mu \cap \nu)(x^{-1}) &= \min\left(\mu(x^{-1}), \nu(x^{-1})\right) \\ &= \min(\mu(x), \nu(x)) \\ &= (\mu \cap \nu)(x)\end{aligned}$$

and the result follows.

On the other hand. the union of two fuzzy subgroups need not be a fuzzy subgroup as shown in the following example.

Example 1.1.3. Let G be the Klein four-group $G = \langle a, b | a^2 = b^2 = (ab)^2 = e \rangle$. Consider the fuzzy subgroups μ an ν of G defined as follows:

$$\mu(e) = \frac{3}{4}, \quad \mu(a) = \frac{5}{8}, \quad \text{and} \quad \mu(b) = \mu(ab) = \frac{3}{8}$$
$$\nu(e) = \frac{7}{8}, \quad \nu(a) = \frac{1}{4}, \quad \text{and} \quad \nu(b) = \frac{5}{8} \quad \text{and} \quad \nu(ab) = \frac{1}{4}.$$

Then

$$(\mu \cup \nu)(ab) = \frac{3}{8} \leq \min((\mu \cup \nu)(a), (\mu \cup \nu)(b)) = \frac{5}{8}.$$

This means that $\nu \cap \nu$ is not a fuzzy subgroup of G.

1.2 Fuzzy Subgroups Induced by Group Homomorphisms

In Section 0.5. some important properties of fuzzy subsets which are induced by functions were stated. In this section, we study the case when the functions are group homomorphisms.

Proposition 1.2.1 [18], [23]. *Let $f : G \to H$ be a group homomorphism and let μ be a fuzzy subgroup of H. Then $f^{-1}(\mu)$ is a fuzzy subgroup of G.*

Proof. Let $g_1, g_2 \in G$, then

$$\begin{aligned} f^{-1}(\mu)(g_1g_2) &= (\mu \circ f)(g_1g_2) = \mu\left(f(g_1g_2)\right) \\ &= \mu\left(f(g_1)f(g_2)\right) \geq \min\left(\mu\left(f(g_1)\right), \mu\left(f(g_2)\right)\right) \\ &= \min\left(f^{-1}(\mu)(g_1), f^{-1}(\mu)(g_2)\right). \end{aligned}$$

Also,

$$\begin{aligned} f^{-1}(\mu)(g^{-1}) &= (\mu \circ f)(g^{-1}) = \mu\left(f(g^{-1})\right) \\ &= \mu\left((f(g))^{-1}\right) = \mu(f(g)) = f^{-1}(\mu)(g) \end{aligned}$$

and the result follows.

The following definition is needed:

Definition 1.2.1 [23]. A fuzzy subset μ of a set X is said to have the *sup property* if for any subset A of X, there is $a_0 \in A$ such that $\mu(a_0) = \sup\{\mu(a) | a \in A\}$.

Proposition 1.2.2 [18], [23]. *Let $f : G \to H$ be a group homomorphism and let μ be a fuzzy subgroup which has the sup property. Then the homomorphic image $f(\mu)$ is a fuzzy subgroup of H.*

Proof. Given $f(x), f(y)$ in $f(G)$, let $x_0 \in f^{-1}(f(x))$, $y_0 \in f^{-1}(f(y))$ be such that

$$\mu(x_0) = \sup_{k \in f^{-1}(f(x))} \mu(k)$$

$$\mu(y_0) = \sup_{k \in f^{-1}(f(y))} \mu(k).$$

Then

$$\begin{aligned} f(\mu)(f(x)f(y)) &= \sup_{z \in f^{-1}(f(x)f(y))} \mu(z) \\ &\geq \min\left(\mu(x_0), \mu(y_0)\right) = \min(f(\mu)(f(x)), f(\mu)(f(y))). \end{aligned}$$

Also,

$$\mu\left((f(x))^{-1}\right) = \sup_{y \in f^{-1}((f(x))^{-1})} \mu(y) \geq \mu(x_0^{-1}) = \mu(x_0) = \mu(f(x))$$

and the result follows.

1.3 Level Subgroups

The notion of level subgroups of a fuzzy subgroup was first introduced by Das [9]. The following result is needed:

Proposition 1.3.1 [9]. *If $t \in I$, e is the identity element of a group G, μ is a fuzzy subgroup of G, and $\mu(e) \geq t$, then the level subset $\mu_t = \{x \in G | \mu(x) \geq t\}$ is a subgroup of G.*

Proof. Since $e \in \mu_t$, therefore $\mu_t \neq \phi$. Now, let $x, y \in \mu_t \Rightarrow \mu(x) \geq t, \mu(y) \geq t$, and $\mu\left(xy^{-1}\right) \geq \min\left\{\mu(x), \mu(y^{-1})\right\} = \min\{\mu(x), \mu(y)\} \geq t \Rightarrow xy^{-1} \in \mu_t$ and the result follows.

Definition 1.3.1 [9]. The subgroup μ_t in Proposition 1.3.1 is called a *level subgroup* of μ.

Proposition 1.3.2 [9]. *Let μ be a fuzzy subgroup of a group G of finite image, $Im(\mu) = \{t_1, t_2, \ldots, t_n\}$ where $t_1 > t_2 \cdots > t_n$. Then the family of subgroups μ_{t_i}, $1 \leq i \leq n$, constitutes all the level subgroups of μ.*

Proof. From the fact that $\mu(e) \geq \mu(x)$ for all $x \in G$, we have $\mu(e) = t_1$. Now, let $s \in I$ and $s \notin \text{Im}(\mu)$. We consider three cases:

(i) $s > t_1 \Rightarrow \mu_s = \emptyset$; otherwise, there is $x \in G$ such that $\mu(x) > s$ which contradicts the fact that $\mu(x) \leq \mu(e) = t_1$.

(ii) $s < t_n \Rightarrow \mu_s \supseteq \mu_{t_n} = G \Rightarrow \mu_s = G = \mu_{t_n}$.

(iii) $t_i < s < t_j$ where $t_i, t_j \in \text{Im}(\mu)$.

Thus by Proposition 1.3.1, either $\mu_{t_i} = \mu_s$ or $\mu_s = \mu_{t_j}$ and the result follows.

The following is an immediate corollary of Proposition 1.3.2.

Corollary 1.3.1. *Two finite fuzzy subgroups of a group G with the same family of level subgroups are equal if and only if they have the same image.*

The following proposition of Das [9] describes an important feature of the family of level subgroups of a given fuzzy subgroup.

Proposition 1.3.3. *The family $\{\mu_t\}_{t \in I}$ of level subgroups of a fuzzy subgroup μ of a group G forms a chain.*

Proof. Let μ_{t_i} and μ_{t_j} be two level subgroups of μ with $t_i > t_j$. Now, $x \in \mu_{t_i} \Rightarrow$

$\mu(x) \geq t_i \Rightarrow \mu(x) \geq t_j \Rightarrow \mu_{t_i} \subseteq \mu_{t_j}$, and the result follows.

The intersection and union of two level subgroups are given in the following proposition:

Proposition 1.3.4. *Let μ and ν be two fuzzy subgroup of a group G, then*

(i) $\mu_t \cap \nu_t = (\mu \cap \nu)_t$.

(ii) $\mu_t \cup \nu_t = (\mu \cup \nu)_t$.

Proof. (i) $x \in \mu_t \cap \nu_t \Leftrightarrow \mu(x) \geq t$ and $\nu(x) \geq t \Leftrightarrow \min(\mu(x), \nu(x)) \geq t \Leftrightarrow x \in (\mu \cap \nu)_t$ which completes the proof of (i).

(ii) $x \in \mu_t \cup \nu_t \Leftrightarrow \mu(x) \geq t$ or $\nu(x) \geq t \Leftrightarrow \max(\mu(x), \nu(x)) \geq t \Leftrightarrow x \in (\mu \cup \nu)_t$ which completes the proof of (ii).

Proposition 1.3.5. *Let μ be a fuzzy subgroup of a group G, with $Im(\mu) = \{t_0, t_1, t_2, \ldots, t_n\}$ where $t_0 > t_1 > t_2 > \cdots > t_n$, then*

(i) If $t_i < t_j$, then $\mu_{t_i} \cap \mu_{t_j} = \mu_{t_j}$ and $\mu_{t_i} \cup \mu_{t_j} = \mu_{t_i}$.

(ii) $\mu_{t_i} = \mu_{t_j}$ if and only if $i = j$.

(iii) $\bigcup_{i=0}^{j} \mu_{t_i} = \mu_{t_j}$ *and* $\bigcap_{i=0}^{j} \mu_{t_i} = \mu_{t_0} = G_\mu$.

Proof. Straightforward.

Proposition 1.3.6 [18]. *Let $f : G \to H$ be a group homomorphism and let μ be a fuzzy subgroup of G. Then* *(i)* $f(\mu_t) \subseteq (f(\mu))_t$.

(ii) $f^{-1}((f(\mu))_t) \subseteq (f^{-1}(f(\mu)))_t$.

(iii) If μ has the sup property, then $(f(\mu))_t \subseteq f(\mu_t)$.

Proof. (i) Let $a \in f(\mu_t) \Rightarrow \exists\ x \in \mu_t$ with $a = f(x)$ such that $\mu(x) \geq t$. Now $f(\mu)(a) = \sup\{\mu(y)|f(y) = a\} \geq \mu(x) \geq t$. Therefore $a \in (f(\mu))_t$. Hence $f(\mu_t) \subseteq (f(\mu))_t$.

(ii) Let $y \in f^{-1}((f(\mu))_t)$. Then $f(y) \in (f(\mu))_t$ where $f(\mu)(f(y)) \geq t$. But $f^{-1}(f(\mu))(y) = f(\mu)f(y) \geq t$. Hence $f^{-1}((f(\mu))_t) \subseteq (f^{-1}(f(\mu))_t$.

(iii) Let $k \in (f(\mu))_t$. i.e.. $f(\mu)(k) \geq t$. Since μ has the sup property, therefore, there exists $x_0 \in G$ such that $\sup\{\mu(x_0)|f(x) = k\} = \mu(x_0)$ and $f(x_0) = h$. Therefore $\mu(x_0) \geq t$ and $f(x_0) \in f(\mu_t)$. Hence $(f(\mu))_t \subseteq f(\mu_t)$. This completes the proof of the theorem.

1.4 The Annuli of a Fuzzy Subgroup

In this section, we introduce the notion of the annuli of a fuzzy subgroup in order to give alternate proofs of some known important properties of fuzzy subgroups.

Henceforward, our interest will be focused on finite groups and hence fuzzy subgroups of finite images and consequently a finite number of level subgroups. So, as an abbreviation, we are going to use the following level-subgroup representation

$$\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} \subseteq \cdots \subseteq \mu_{t_n} = G$$

to mean that μ is a fuzzy subgroup of G with finite image $\mathrm{Im}(\mu) = \{t_0, t_1, \ldots, t_n\}$ where each $t_i \in I$ and $t_0 > t_1 > \cdots t_n$ and whose level subgroups are $G_\mu = \mu_{t_0}$, $\mu_{t_1}, \mu_{t_2} \ldots, \mu_{t_n} = G$.

Definition 1.4.1. Consider the fuzzy subgroup

$$\mu : G_\mu = \mu_{t_0} \subsetneq \mu_{t_1} \subsetneq \cdots \subsetneq \mu_{t_n} = G.$$

Then the set

$$A_\mu^{t_i} = \begin{cases} \mu_{t_i} - \mu_{t_{i-1}}, & \text{if } \ 1 \leq i \leq n \\ G_\mu = \mu_{t_0}, & \text{if } \ i = 0, \end{cases}$$

is called the t_i-*annulus* of μ.

From Definition 1.4.1. we get immediately

(i) $x \in A_\mu^{t_i}, \quad 0 \leq i \leq n \Leftrightarrow \mu(x) = t_i$.

(ii) $\bigcup_{i=0}^{k} A_\mu^{t_i} = \mu_k, \quad 0 \leq k \leq n$, and

(iii) the set $\left\{A_\mu^{t_i}\right\}$ of all annuli of μ partition G into a family of mutually disjoint nonempty subsets, i.e.

(1) $A_\mu^{t_i} \cap A_\mu^{t_j} = \emptyset \quad$ for $\ i \neq j$.

(2) $\bigcup_{i=0}^{n} A_\mu^{t_i} = G$.

Example 1.4.1. For the fuzzy subgroup μ of D_4 given in Example 1.1.1, we have

$$\mu : G_\mu = \mu_{3/4} \subset \mu_{1/2} \subset \mu_{1/3} = G = D_4$$

where

$$\mu_{3/4} = \{e, g\}, \ \mu_{1/2} = \left\{e, g, f^2, f^2 g\right\}$$

and

$$\mu_{1/3} = D_4 = \left\{e, f, f^2, f^3, g, fg, f^2 g, f^3 g\right\}.$$

The annuli of μ are

$$A^{3/4} = \{e, g\}, \ A^{1/2} = \left\{f^2, f^2 g\right\} \quad \text{and} \quad A^{1/3} = \left\{f, f^3, fg, f^3 g\right\}.$$

The annuli of a fuzzy subgroup can be used to justify the definition of a fuzzy subgroup as follows:

Let $x, y \in G \Rightarrow x \in A_\mu^{t_i}$ and $y \in A_\mu^{t_j}$ for some i and j. Thus $\mu(x) = t_i$ and $\mu(y) = t_j$. Suppose that $i < j \Rightarrow t_i > t_j \Rightarrow \mu_{t_i} \subseteq \mu_{t_j}$ and hence

$$x, y \in \mu_{t_j} \Rightarrow \mu(xy) \geq t_j \Rightarrow \mu(xy) \geq \min(\mu(x), \mu(y)).$$

Now, $x \in A_\mu^{t_i} \Rightarrow x^{-1} \in \mu_{t_i}$. Suppose $\mu(x^{-1}) > t_i$. This means that $x^{-1} \in A_\mu^{t_j}$ for $j < i \Rightarrow x^{-1} \in \mu_{t_j} \Rightarrow x \in \mu_{t_j}$ and hence $\mu(x) \neq t_i$. This contradiction gives $\mu(x^{-1}) = t_i$ and the clarification is completed.

The basic properties of fuzzy subgroups can now be proved using an annulus approach as follows:

Proposition 1.4.1. *Let $\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} \subseteq \cdots \subseteq \mu_{t_n} = G$, then $\mu(x^k) \geq \mu(x)$ for all $x \in G$.*

Proof. Let $x \in G$, then $x \in A_\mu^{t_i}, \mu(x) = t_i$ for some i, $0 \leq i \leq n$ and hence $x \in \mu_{t_i}$. Thus $x^n \in \mu_{t_i}$ which means $\mu(x^n) \geq t_i$, i.e., $\mu(x^n) \geq \mu(x)$ and the result follows.

Proposition 1.4.2. *If $x \in A_\mu^{t_i}$ and $y \in A_\mu^{t_j}$ with $i < j$, then xy and $yx \in A_\mu^{t_j}$. In other words. $\mu(x) > \mu(y) \Rightarrow \mu(xy) = \mu(yx) = \mu(y)$.*

Proof. $i < j \Rightarrow t_i > t_j \Rightarrow \mu_{t_i} \subset \mu_{t_j} \Rightarrow x \in \bigcup_{k=0}^{j-1} A_\mu^{t_k}$. Suppose that $xy \in \bigcup_{k=0}^{j-1} A_\mu^{t_k} \Rightarrow$ $x^{-1}(xy) = y \in \bigcup_{k=0}^{j-1} A_\mu^{t_k}$ which contradicts the assumption that $y \in A_\mu^{t_j}$. Thus $xy \in A_\mu^{t_j}$. Similarly, we prove that $yx \in A_\mu^{t_j}$, and the result follows.

The converse of Proposition 2.2.1 is true after adding one more condition as shown in the following proposition.

Proposition 1.4.3. *If y, xy and $yx \in A_\mu^{t_j}$, then $x \in A_\mu^{t_i}$ where $t_i \geq t_j$. In other words, if $\mu(xy) = \mu(yx) = \mu(y)$, then $\mu(x) \geq \mu(y)$.*

Proof. xy and $y \in A_\mu^{t_j} \Rightarrow (xy)y^{-1} \in \mu_{t_j} \Rightarrow x \in \mu_{t_j} \Rightarrow x \in A_\mu^{t_i}$ where $t_i \geq t_j$ and the result follows.

Proposition 1.4.4. *y and xy lie in the same annulus for all $y \in G$ if and only if $x \in G_\mu$. In other words. $\mu(xy) = \mu(y)$ for every $y \in G$ if and only if $\mu(x) = \mu(e)$.*

Proof ($\Rightarrow$) Let y and xy lie in the same annulus for all $y \in G$. Thus e and xe lie in the same annulus which means $x \in G_\mu$.

($\Leftarrow$) Let $x \in G_\mu$ and let $y \in G$ if $y \in G_\mu \Rightarrow xy \in G_\mu$, while if $y \in A_\mu^{t_i}$ with $i > 0$ and $x \in A_\mu^{t_0}$. then $y, xy \in A_\mu^{t_i}$ by Proposition 1.4.2. Thus, in any case, y, xy lie on the same annulus. and the result follows.

Proposition 1.4.5. *Consider the fuzzy subgroup*

$$\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \cdots \subseteq \mu_{t_n} = G.$$

If for some $x, y \in G$ we have $xy \in G_\mu = \mu_{t_0}$, then x and y lie in the same annulus. In other words, $\mu(xy) = \mu(e) \Rightarrow \mu(x) = \mu(y)$.

Proof. Let $x \in A_\mu^{t_i}$ and $y \in A_\mu^{t_j}$ with $t_i \geq t_j$. Suppose $t_i > t_j$, then $xy \in A_\mu^{t_j}$ (by Proposition 1.4.2) which contradicts the assumption $xy \in A_\mu^{t_0}$. Hence $t_i = t_j$ which completes the proof.

The converse of the above proposition is not true in general. Consider the following example.

Example 1.4.2. Consider the fuzzy subgroup

$$\mu : \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} = G = Z_8 \quad (t_0 > t_1 > t_2)$$

where

$$A_\mu^{t_0} = \{0, 4\}, \ A_\mu^{t_1} = \{2, 6\}, \ A_\mu^{t_2} = \{1, 3, 5, 7\}.$$

Now $1 \in A_\mu^{t_2}$ and $5 \in A_\mu^{t_2}$ but $1 + 5 = 6 \notin A_\mu^{t_0}$.

Chapter Two

An Annulus Approach to Fuzzy Normal Subgroups

In this chapter, we study the various equivalent statements on fuzzy normal subgroups and straighten out some results on fuzzy normal and fuzzy abelian groups. We also introduce the notion of the centralizer of a fuzzy subgroup.

2.1 Fuzzy Normal Subgroups: Definition and Examples

The concept of fuzzy normal subgroups was introduced by Lieu [17] and Mukherjee and Bhattacharya [19] as a generalization of the concept of normal subgroups.

Definition 2.1.1 [17], [19]. A fuzzy subgroup μ of a group G is said to be fuzzy normal if $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

Proposition 2.1.1. *If μ is a fuzzy normal subgroup of a group G, then the subgroup $G_\mu = \{x \in G \,|\, \mu(x) = \mu(e)\}$ is normal in G.*

Proof. For any $g \in G_\mu$ and $x \in G$, we have $\mu\,(x^{-1}gx) = \mu\,((gx)x^{-1}) = \mu(g) = \mu(e)$. Thus $x^{-1}gx \in G_\mu$ and the result follows.

Proposition 2.1.2. *Let μ and ν be the fuzzy normal subgroups of a group G, then $\mu \cap \nu$ is also a fuzzy normal subgroup of G.*

Proof. Let x and y be any two elements of G, then we have

$$\begin{aligned}(\mu \cap \nu)(xy) &= \min(\mu(xy), \nu(xy)) \\ &= \min(\mu(yx), \nu(yx)) \\ &= (\mu \cap \nu)(yx),\end{aligned}$$

and the result follows.

Corollary 2.1.1. *The intersection of any number of fuzzy normal subgroups of a group G is a fuzzy normal subgroup.*

Proposition 2.1.3. *Let $f : G \to H$ be a group homomorphism and let μ and ν be two fuzzy normal subgroups of G and H respectively. Then*

(i) $f^{-1}(\nu)$ is a fuzzy normal subgroup of G.

(ii) If f is onto, then $f(\mu)$ is a fuzzy normal subgroup of H.

Proof. (i) Let $x, y \in G$, then

$$\begin{aligned}f^{-1}(\nu)(xy) &= \nu(f(xy)) = \nu(f(x)f(y)) \\ &= \nu(f(y)f(x)) = \nu(f(yx)) \\ &= f^{-1}(\nu)(yx).\end{aligned}$$

Thus $f^{-1}(\nu)$ is a fuzzy normal subgroup of H.

(ii) Since f is onto, therefore $f(\mu)$ is a fuzzy subgroup of G. Now let $x, y \in H$, then

$$\begin{aligned} f(\mu)(xy) &= \sup\{\mu(g) | g \in G \quad \text{and} \quad f(g) = xy\} \\ &= \sup\{\mu(g_1 g_2) | g_1, g_2 \in G \quad \text{with} \quad f(g_1) = x,\ f(g_2) = y\} \\ &= \sup\{\mu(g_2 g_1) | g_1, g_2 \in G \quad \text{with} \quad f(g_1) = x,\ f(g_2) = y\} \\ &= f(\mu)(yx). \end{aligned}$$

Hence $f(\mu)$ is a fuzzy normal subgroup of H. This completes the proof of the theorem.

Proposition 2.1.4 [19]. *Any level subgroup μ_t of a fuzzy normal subgroup μ of a group G is normal in G.*

Proof. We have $\mu_t = \{x \in G | \mu(x) \geq t \quad \text{where} \quad \mu(e) \geq t\}$. Thus for any $g \in \mu_t$ and for any $x \in G$, we get $\mu(x^{-1}gx) = \mu((gx)x^{-1}) = \mu(g) \geq t$, and the result follows.

Now we give an alternate proof of the following known proposition [19] using the annulus approach.

Proposition 2.1.5. *Consider the following fuzzy subgroup*

$$\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} \subseteq \cdots \subseteq \mu_{t_n} = G.$$

If μ_{t_i} is normal in G for all $0 \leq i \leq n-1$, then μ is a fuzzy normal subgroup of G.

Proof. Let x and y be any two elements of G, then $xy \in A_\mu^{t_i}$ for some $0 \leq i \leq n$, which means $xy \in \mu_{t_i}$ and since μ_{t_i} is normal in G, therefore $x^{-1}(xy)x = yx \in \mu_{t_i}$. Now, suppose that $yx \in A_\mu^{t_k}$ for some $t_k > t_i$. Thus, $yx \in \mu_{t_k} \subseteq \mu_{t_i}$. Since μ_{t_k} is normal in G, therefore $x(yx)x^{-1} = xy \in \mu_{t_k}$. Thus $xy \notin A_\mu^{t_i}$ which is a contradiction to the assumption. Hence $yx \in A_\mu^{t_i}$, and consequently $\mu(xy) = \mu(yx)$, i.e., μ is a fuzzy normal subgroup of G. This completes the proof.

The above results can now be summarized in the following theorem:

Theorem 2.1.1. *For the fuzzy subgroup*

$$\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \cdots \subseteq \mu_{t_n} = G,$$

the following statements are equivalent:

1. *μ is a fuzzy normal subgroup of G.*
2. *μ_{t_i} is normal in G for all $0 \leq i \leq n-1$.*
3. *For any elements x and y in G, xy and yx lie in the same annulus of μ.*

Now, we give an example of a fuzzy normal subgroup.

Example 2.1.1. Let $G = D_4$ be the dihedral group of order 8, i.e.,

$$D_4 = \langle a, b | a^4 = b^2 = e \quad \text{and} \quad a^i b = ba^{-i}, \quad 1 \leq i \leq 3\}.$$

Consider the fuzzy subgroup μ of D_4:

$$\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} = D_4$$

where

$$\mu_{t_0} = \left\{e, a^2\right\}, \ \mu_{t_1} = \left\{e, a^2, b, a^2b\right\}, \ \mu_{t_2} = D_4,$$

then μ is a fuzzy normal subgroup of D_4 and this can be easily checked from the following table:

μ	e	a	a^2	a^3	b	ab	a^2b	a^3b
e	t_0	t_2	t_0	t_2	t_1	t_2	t_1	t_2
a	t_2	t_0	t_2	t_0	t_2	t_1	t_2	t_1
a^2	t_0	t_2	t_0	t_2	t_1	t_2	t_1	t_2
a^3	t_2	t_0	t_2	t_0	t_2	t_1	t_2	t_1
b	t_1	t_2	t_1	t_2	t_0	t_2	t_0	t_2
ab	t_2	t_1	t_2	t_1	t_2	t_0	t_2	t_0
a^2b	t_1	t_2	t_1	t_2	t_0	t_2	t_0	t_2
a^3b	t_2	t_1	t_2	t_1	t_2	t_0	t_2	t_0

It is clear that μ_{t_0} and μ_{t_1} are both normal in $\mu_{t_2} = D_4$.

2.2 An Annulus Approach to More Equivalent Statements

In this section, we use the annuli of a fuzzy subgroup to prove the equivalent statement of fuzzy normal subgroups known in the literature. Other proofs can be found in [7], [19], [20]. Then we give an equivalent statement which straightens out an incorrect statement given in [7].

Proposition 2.2.1. *A fuzzy subgroup*

$$\mu : G_\mu = \mu_{t_0} \subsetneq \mu_{t_1} \subsetneq \mu_{t_2} \subsetneq \cdots \subsetneq \mu_{t_n} = G$$

is fuzzy normal if and only if for any x, y in G, x and $y^{-1}xy$ lie in the same annulus.

Proof. ($\Rightarrow$) Let μ be fuzzy normal, i.e., μ_{t_i} is normal in G for all $0 \leq i \leq n$. Let x and y be any two elements of G, then $x \in A_\mu^{t_i}$ and $y^{-1}xy \in A_\mu^{t_k}$ for some i and k. But since μ_{t_i} is normal in G, therefore $t_k \geq t_i$. Now suppose $t_k > t_i$, then $y^{-1}xy \in \mu_{t_k}$ and since μ_{t_k} is normal in G, therefore $y\,(y^{-1}xy)\,y^{-1} = x \in \mu_{t_k}$. This means $x \notin A_\mu^{t_i}$. Thus $t_k = t_i$ and hence x and $y^{-1}xy$ lie in the same annulus.

($\Leftarrow$) Let x and y be any two elements of G and let x and $y^{-1}xy$ lie in the same annulus $A_\mu^{t_k}$ for some k. Therefore, for any $x \in \bigcup_{i=1}^{i} A_\mu^{t_k} = \mu_{t_k}$, x and $y^{-1}xy$ lie in the same annulus $A_\mu^{t_k}$ for some $0 \leq k \leq i$ and for any $y \in G$. Therefore, μ_{t_i} is normal in G for all $0 \leq i \leq n$ and hence μ is a fuzzy normal subgroup of G. This completes the proof of the proposition.

The following corollary is an immediate consequence of Proposition 2.2.1

Corollary 2.2.1. *A fuzzy subgroup*

$$\mu : G_\mu = \mu_{t_0} \subsetneq \mu_{t_1} \subsetneq \cdots \subsetneq \mu_{t_n} = G$$

is fuzzy normal if and only if $\mu\,(y^{-1}xy) \geq \mu(x)$.

Proposition 2.2.2. *Consider the fuzzy subgroup*

$$\mu : G_\mu = \mu_{t_0} \subsetneq \mu_{t_1} \subsetneq \cdots \subsetneq \mu_{t_n} = G$$

and let $[x, y] = x^{-1}y^{-1}xy$ *be the commutator of* x *and* y *in* G. *Then* μ *is fuzzy normal if and only if whenever* $[x, y] \in A_\mu^{t_i}$, $x \in A_\mu^{t_j}$ *such that* $t_i \geq t_j$ $\forall x, y \in G$.

Proof. ($\Rightarrow$) Let μ be fuzzy normal, and suppose that $x \in A_\mu^{t_j}$, then x, x^{-1} and $y^{-1}xy$ lie in the same annulus $A_\mu^{t_j}$ for all $y \in G$ (Proposition 2.2.1). Thus, $x^{-1}(y^{-1}xy) = x^{-1}y^{-1}xy = [x, y]$ lie in $A_\mu^{t_i}$ where $t_i \geq t_j$ according to the definition of fuzzy subgroups.

($\Leftarrow$) Let $[x, y] \in A_\mu^{t_i}$ and $x \in A_\mu^{t_j}$ such that $t_i \geq t_j$, thus the product $x[x, y] \in A_\mu^{t_j}$ (Proposition 1.4.2), i.e., $x(x^{-1}y^{-1}xy) = y^{-1}xy \in A_\mu^{t_j}$ for all $y \in G$. Thus μ is fuzzy normal which completes the proof.

Bhattacharya and Mukherjee [7, Proposition 4.1] give the following statement:

> "A fuzzy subgroup μ of a group G is fuzzy normal if and only if $\mu([x, y]) = \mu(e)$."

One direction of this statement appears to be incorrect as shown in the following counter-example:

Example 2.2.1. Consider the fuzzy normal subgroup

$$\mu : G_\mu = \mu_{t_0} = \mu_{t_0} \subsetneq \mu_{t_1} \subsetneq \mu_{t_2} = G = D_4$$

where

$$D_4 = \left\langle a, b | a^4 = b^2 = e,\ a^i b = ba^{-1} \right\rangle$$

is the dihedral group of order 8 and

$$A_\mu^{t_0} = \{e\},\ A_\mu^{t_1} = \left\{a^2\right\},\ A_\mu^{t_2} = \left\{b, a^2 b\right\},\ A_\mu^{t_3} = \left\{a, a^3, ab, a^3 b\right\}.$$

Now

$$[a, b] = a^{-1}bab = (a^3)(a^3 b)(b) = a^2$$

i.e.,

$$[a, b] \in A_\mu^{t_1},$$

which means

$$\mu([a, b]) \neq \mu(e).$$

Now we give a correct version of the above statement.

Proposition 2.2.3. *A fuzzy subgroup*

$$\mu : G_\mu = \mu_{t_0} \subsetneq \mu_{t_1} \subsetneq \cdots \subsetneq \mu_{t_n} = G$$

is fuzzy normal if and only if x and $[x, y]x$ lie in the same annulus.

Proof. μ is fuzzy normal $\Leftrightarrow$ x and $x^{-1}(y^{-1}xy)x$ lie in the same annulus $\forall\, x, y \in G$ $\Leftrightarrow$ x and $(x^{-1}y^{-1}xy)\,x$ lie in the same annulus $\forall\, x, y \in G \Leftrightarrow x$ and $[x, y]x$ lie in the same annulus. This completes the proof.

The results in Sections 2.1 and 2.2 can now be stated in one theorem:

Theorem 2.2.1. *Given the fuzzy subgroup*

$$\mu : G_\mu = \mu_{t_0} \subsetneq \mu_{t_1} \subsetneq \mu_{t_1} \subsetneq \mu_{t_n} = G,$$

the following statements are equivalent:

(A) μ *is fuzzy normal.*

(B) xy *and* yx *lie in the same annulus for all* $x, y \in G$.

(C) μ_{t_i} *is normal in* G *for all* $0 \leq i \leq n-1$.

(D) x *and* $y^{-1}xy$ *lie in the same annulus for all* $x, y \in G$.

(E) If $y^{-1}xy \in A_{\mu}^{t_i}$, *then* $x \in A_{\mu}^{t_j}$ *for some* $t_i \geq t_j$.

(F) x *and* $[x, y]x$ *lie in the same annulus for all* x *and* $y \in G$.

(G) Whenever $[x, y] \in A_{\mu}^{t_i}$, $x \in A_{\mu}^{t_j}$ *such that* $t_i \geq t_j \quad \forall x, y \in G$.

2.3 Fuzzy Abelian Groups

It is clear that every fuzzy subgroup of an abelian group is a fuzzy normal subgroup. The converse of this statement is not necessarily true as shown in Example 2.1.1 in which μ is fuzzy normal while G is a nonabelian group.

In group theory the following statements are equivalent for a group G:

(A) G is an abelian group.

(B) $[x, y] = e$ for all $x, y \in G$.

(C) $G' = \langle e \rangle$, where G' is the derived group of G.

This motivated Mukherjee and Bhattacharya [7] to define fuzzy abelian subgroups as follows:

Definition 2.3.1 [7] A fuzzy subgroup μ of a group G is fuzzy abelian if

$$\mu([x, y]) = \mu(e) \quad \text{for every} \quad x, y \in G.$$

Lemma 2.3.1. *If μ is a fuzzy abelian group of a group G, then:*

(i) $G' \subseteq G_\mu$.

(ii) G_μ *is normal in* G.

(iii) G/G_μ *is abelian.*

Proof. Straightforward.

The assertion given by Bhattacharya and Mukherjee [7, p. 831] that a "fuzzy subgroup μ of a finite group G is fuzzy abelian if and only if G_μ is abelian subgroup in G" is false. The following is a counter-example:

Example 2.3.1. Let G be any finite group and let μ be defined by $\mu(x) = \mu(e)$ for all $x \in G$. Clearly, μ is a fuzzy abelian group while $G_\mu = G$ is not necessarily an abelian group.

A correct version of the assertion stated above is given in the following proposition.

Proposition 2.3.2. *A fuzzy normal subgroup μ of a group G is fuzzy abelian if* *and only if G/G_μ is an abelian group.*

Proof. ($\Rightarrow$) If μ be fuzzy abelian, then $G' \subseteq G_\mu$ and hence G/G_μ is abelian.

($\Leftarrow$) If G/G_μ is abelian, then $G_\mu \supseteq G'$ and hence μ is fuzzy abelian. This completes the proof.

Bhattacharya and Mukherjee's wrong assertion leads to another false statement [7, Proposition 4.5], *viz.*, "Let μ and ν be two fuzzy subgroups of a group G such that $\nu \leq \mu$, $\nu(e) = \mu(e)$, and μ is fuzzy abelian. Then ν is fuzzy abelian." The following is a counter-example:

Example 2.3.2. Let μ and ν be two fuzzy subgroups of D_4 defined as follows:

$$\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} = G = D_4$$

and

$$\nu : G_\nu = \nu_{t_0} \subseteq \nu_{t_3} \subseteq \nu_{t_4} \subseteq \nu_{t_4} = D_4$$

where $t_0 > t_1 > t_2 > t_3 > t_4 > t_5$, $A_\mu^{t_0} = \{e, a^2\}$, $A_\mu^{t_1} = \{a, a^3\}$, $A_\mu^{t_2} = \{b, ab, a^2b, a^3b\}$, $A_\nu^{t_0} = \{e\}$, $A_\nu^{t_3} = \{b\}$, $A_\nu^{t_4} = \{a^3, a^3b\}$, and $A^{t_5} = \{a, a^2, ab, a^2b\}$.

It is easy to check that μ is fuzzy abelian, $\mu(e) = \nu(e)$ and $\nu \leq \mu$ even if ν is not fuzzy abelian.

2.4 The Normalizer and the Centralizer of a Fuzzy Subgroup

The notion of the normalizer of a fuzzy subgroup μ of a group G was introduced by Mukherjee and Bhattacharya in [20] as follows:

Definition 2.4.1. The normalizer of μ is the set given by

$$N(\mu) = \{g \in G \mid \mu(g^{-1} x g) = \mu(x), \quad \forall x \in G\}.$$

Proposition 2.4.1 [20]. *Let μ be a fuzzy subgroup of G, then*

(i) $N(\mu)$ is a subgroup of G.

(ii) μ is fuzzy normal $\Leftrightarrow N(\mu) = G$.

Proof. (i) Let g_1, g_2 be any two elements of $N(\mu)$. Therefore for any $x \in G$, we have

$$\begin{aligned} \mu\left(((g_1 g_2^{-1})^{-1} x (g_1 g_2^{-1})\right) &= \mu\left(g_2 g_1^{-1} x g_1 g_2^{-1}\right) \\ &= \mu\left(g_1 x g_1^{-1}\right), \qquad \text{since} \quad g_2 \in N(\mu) \\ &= \mu(x). \end{aligned}$$

Thus $g_1 g_2^{-1} \in N(\mu)$, and hence $N(\mu)$ is a subgroup of G.

(ii) ($\Rightarrow$) Let μ be fuzzy normal, then $\mu(g^{-1} x g) = \mu(x)$ for any $x, g \in G$ and hence $N(\mu) = G$.

($\Leftarrow$) If $N(\mu) = G$, then for any $x, y \in G$, we have

$$\begin{aligned} \mu(xy) &= \mu(xyxx^{-1}) = \mu\left(x(yx)x^{-1}\right) \\ &= \mu(yx) \text{ (since } x \in N(\mu)). \end{aligned}$$

Hence μ is fuzzy normal.

Now we introduce the centralizer $C(\mu)$ of a fuzzy subgroup μ of G as follows:

Definition 2.4.2. The centralizer of μ is the set given by

$$C(\mu) = \{g \in G | \mu([g, x]) = \mu(e), \quad \forall\, x \in G\}.$$

Proposition 2.4.2. *Let μ be a fuzzy subgroup of a group G and let $C(G)$ denote the center of G. Then*

(i) $C(G)$ is a subset of $C(\mu)$.

(ii) If μ is a fuzzy normal subgroup of G, then $C(\mu)$ is a normal subgroup of G.

Proof (i) Let $g \in C(G) \Rightarrow [g, x] = e \quad \forall\, x \in G \Rightarrow \mu([g, x]) = \mu(e) \quad \forall\, x \in G \Rightarrow g \in C(\mu)$ and hence $C(G)$ is a subset of $C(\mu)$.

(ii) Let μ be a fuzzy normal subgroup of G and let $g_1, g_2 \in C(\mu)$. Thus for every $x \in G$, we have

$$\begin{aligned}
\mu\left([g_1 g_2^{-1}, x]\right) &= \mu\left(g_2 g_1^{-1} x^{-1} g_1 g_2^{-1} x\right) \\
&= \mu\left(g_2 g_1^{-1} x^{-1} g_1 x \, x^{-1} g_2^{-1} x g_2 g_2^{-1}\right) \\
&= \mu\left(g_2 [g_1, x] \, [x, g_2] g_2^{-1}\right) \\
&= \mu\left([g_1, x] \, [x, g_2]\right) \qquad \text{since } \mu \text{ is fuzzy normal} \\
&\geq \min\left(\mu([g_1, x]), \mu([x, g_2])\right) \\
&\geq \min\left(\mu(e), \mu(e)\right) \qquad \text{since } g_1, g_2 \in C(\mu) \\
&= \mu(e).
\end{aligned}$$

Thus

$$\mu\left([g_1 g_2^{-1}, x]\right) = \mu(e) \quad \forall\, g_1, g_2 \in C(\mu),$$

and hence $C(\mu)$ is a subgroup of G.

It remains to show that $C(\mu)$ is normal in G. Let $g \in C(\mu)$ and $x, y \in G$, then

$$\begin{aligned}
\mu\left([x^{-1}gx.y]\right) &= \mu\left(x^{-1}g^{-1}xy^{-1}x^{-1}gxy\right) \\
&= \mu\left((x^{-1}g^{-1}xg)(g^{-1}y^{-1}gy)(y^{-1}g^{-1}x^{-1})(gxy)\right) \\
&= \mu\left([x.g]\;[g,y]\,y^{-1}[g,x]y\right) \\
&\geq \min\left(\mu[x,g]\right),\ \mu\left([g,y]y^{-1}[g,x]y\right) \\
&= \mu\left([g,y]\,y^{-1}[g,x]y\right) \qquad \text{since} \ \ \mu[g,y] = \mu(e) \\
&\geq \min\left(\mu[g,y],\right),\ \mu\left(y^{-1}[g,x]y\right) \\
&= \mu\left(y^{-1}[g,x]y]\right) \qquad \text{since} \ \ \mu[g,y] = \mu(e) \\
&= \mu([g.x]) \qquad \text{since} \ \ \mu \ \ \text{is fuzzy normal} \\
&= \mu(e) \qquad \text{since} \ \ g \in C(\mu),
\end{aligned}$$

and hence $\mu\left([x^{-1}gx.y]\right) = \mu(e)$. Therefore $x^{-1}gx \in C(\mu)$ for all $x \in G$. Thus $C(\mu)$ is a normal subgroup of G. This completes the proof.

Chapter Three

An Annulus Approach to Fuzzy Cosets

In this chapter. we study the properties of the left and right fuzzy cosets determined by a fuzzy subgroup. describe the structure of each annulus of a fuzzy coset. and give a necessary and sufficient condition for an annulus of a fuzzy coset to be a subgroup.

3.1 Fuzzy Cosets: Definition and Examples

In [19]. Mukherjee and Bhattacharya introduced the notion of fuzzy cosets as follows:

Definition 3.1.1. Let μ be a fuzzy subgroup of a group G. The left fuzzy coset ${}_x\mu$ and the right fuzzy cost μ_x of μ determined by $x \in G$ are the fuzzy subsets of G defined, for all $g \in G$. as follows:

$$ {}_x\mu(g) = \mu(x^{-1}g)$$

and

$$\mu_x(g) = \mu(gx^{-1}).$$

It is clear that the images of μ. ${}_x\mu$ and μ_x are equal for every $x \in G$. Thus if $t \in \mathrm{Im}(\mu)$. then the t-annuli $A^t_{{}_x\mu}$ and $A^t_{\mu_x}$ of ${}_x\mu$ and μ_x, respectively, are given by

$$A^t_{{}_x\mu} = \left\{g \in G|{}_x\mu(g) = \mu(x^{-1}g) = t\right\}$$

and

$$A^t_{\mu_x} = \left\{ g \in G \mid \mu_x(g) = \mu(gx^{-1}) = t \right\}.$$

Example 3.1.1. Consider the fuzzy subgroup

$$\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} = G = D_4$$

where

$$D_4 = \langle a, b \mid a^4 = b^2,\ a^i b = b a^{-i} \rangle$$

is the dihedral group of order 8, and

$$A^{t_0}_\mu = \{e, b\},\ A^{t_1}_\mu = \{a^2, a^2 b\}, \quad \text{and} \quad A^{t_2}_\mu = \{a, a^3, ab, a^3 b\}.$$

Then μ and all of its left and right cosets can be presented by the following tables:

(i) The left fuzzy cosets

$\mu = {}_e\mu = {}_b\mu$:

t_0-annulus	t_1-annulus	t_2-annulus
e	a^2	a, ab
b	$a^2 b$	a^3, $a^3 b$

${}_a\mu = {}_{ab}\mu$:

t_0-annulus	t_1-annulus	t_2-annulus
a	a^3	e, b
ab	$a^3 b$	a^2, $a^2 b$

${}_{a^3}\mu = {}_{a^3 b}\mu$:

t_0-annulus	t_1-annulus	t_2-annulus
a^3	a	e, b
$a^3 b$	ab	a^2, $a^2 b$

${}_{a^2}\mu = {}_{a^2b}\mu$:

t_0-annulus	t_1-annulus	t_2-annulus
a^2	a	e ab
a^2b	ab	a^3 a^3b

(ii) <u>The right fuzzy cosets</u>

$\mu = \mu_e = \mu_b$:

t_0-annulus	t_1-annulus	t_2-annulus
e	a^2	a ab
b	a^2b	a^3 a^3b

$\mu_a = \mu_{a^3b}$:

t_0-annulus	t_1-annulus	t_2-annulus
a	a^3	e b
a^3b	ab	a^2 a^2b

$\mu_{a^2} = \mu_{a^2b}$:

t_0-annulus	t_1-annulus	t_2-annulus
a^2	a	e b
a^2b	b	a^3 a^3b

$\mu_{a^3} = \mu_{ab}$:

t_0-annulus	t_1-annulus	t_2-annulus
a^3	a	e b
ab	a^3b	a^2 a^2b

The following two remarks can be easily drawn from the above example:

Remark 1. The two sets of the left fuzzy cosets and the right fuzzy cosets are not equal (e.g., $\mu_a \neq {}_a\mu$).

Remark 2. Contrary to what is known in group theory, the intersection of two left (right) fuzzy cosets of a fuzzy subgroup is not necessarily empty (e.g., ${}_a\mu \cap {}_{a^3}\mu \neq \phi$).

Proposition 3.1.1. *If μ is a fuzzy normal subgroup of a group G, then $\mu_x = {}_x\mu$ for every $x \in G$.*

Proof.
$$\begin{aligned} \mu_x(g) &= \mu(gx^{-1}) = \mu(x^{-1}g) \quad (\mu \text{ is fuzzy normal}) \\ &= {}_x\mu(g), \end{aligned}$$

and the result follows.

3.2 Properties of Fuzzy Cosets

In what follows μ is a fuzzy subgroup of a group G.

Proposition 3.2.1. *$\mu_x = \mu$ if and only if $x \in G_\mu$.*

Proof ($\Rightarrow$) Let $\mu_x = \mu$, then $\mu_x(y) = \mu(y)$ for every $y \in G$. Thus, in particular, $\mu_x(x) = \mu(x)$, i.e., $\mu(xx^{-1}) = \mu(x) \Rightarrow \mu(e) = \mu(x) \Rightarrow x \in G_\mu$.

($\Leftarrow$) Let $x \in G_\mu$, then for every $g \in G$, we have

$$\mu_x(g) = \mu(gx^{-1}) = \mu(g) \quad \text{(Proposition 1.4.4)}.$$

Hence $\mu_x = \mu$ which completes the proof.

Proposition 3.2.2. $\mu_g = \mu_h$ *if and only if* $gh^{-1} \in G_\mu$.

Proof. $(\Rightarrow)$ Let $\mu_g = \mu_h \Rightarrow \mu_g(g) = \mu_h(g) \Rightarrow \mu(e) = \mu(gh^{-1}) \Rightarrow gh^{-1} \in G_\mu$.

$(\Leftarrow)$ Let $gh^{-1} \in G_\mu$, then

$$\begin{aligned} \mu_g(x) = \mu(xg^{-1}) &= \mu\left((xg^{-1})(gh^{-1})\right) \quad \text{(Proposition 1.4.4.)} \\ &= \mu(xh^{-1}) = \mu_h(x). \end{aligned}$$

This completes the proof.

Proposition 3.2.3. $\mu_g(x) = \mu_x(g)$.

Proof.
$$\begin{aligned} \mu_g(x) &= \mu(xg^{-1}) = \mu\left((xg^{-1})^{-1}\right) \\ &= \mu(gx^{-1}) = \mu_x(g). \end{aligned}$$

This completes the proof.

The following proposition gives the structure of the t-annulus of the right fuzzy coset μ_x in terms of the t-annulus of μ.

Proposition 3.2.4. $A^t_{\mu_x} = A^t_\mu x$.

Proof. $g \in A^t_{\mu_x} \Leftrightarrow \mu_x(g) = t \Leftrightarrow \mu(gx^{-1}) = t$

$\Leftrightarrow gx^{-1} \in A^t_\mu \Leftrightarrow g \in A^t_\mu x$.

This completes the proof.

The above results remain valid if μ_x is replaced by ${}_x\mu$. In such a case, $A^t_{\mu_x} = xA^t_\mu$.

3.3. The Subgroups G_{μ_x} and G_{x^μ}

Let μ be a fuzzy subgroup of a group G. The subsets G_{μ_x} and G_{x^μ} are defined as follows:

$$G_{\mu_x} = \left\{g \in G | \mu_x(g) = \mu(gx^{-1}) = \mu(x)\right\}$$

and

$$G_{x^\mu} = \left\{g \in G | \ {}_x\mu(g) = \mu(x^{-1}g) = \mu(x)\right\}.$$

Example 3.1.1 can be used to verify that the subsets G_{μ_x} and G_{x^μ} are subgroups of G for all $x \in G$. But this is not always true as shown in the following example:

Example 3.3.1. Consider the fuzzy subgroup

$$\mu : G_\mu = \mu_{t_0} \subseteq \mu_{t_1} \subseteq \mu_{t_2} = D_4 = G$$

where

$$A_\mu^{t_0} = \{e\}, \ A_\mu^{t_1} = \left\{a, a^2, a^3\right\}$$

and

$$A_\mu^{t_2} = \left\{b, ab, a^2b, a^3b\right\}.$$

Then the right fuzzy cosets μ_a of μ can be represented by the following table.

μ_a :

t_0-annulus	t_1-annulus	t_2-annulus
a	e, a^2, a^3	b, ab, a^2b, a^3b

Thus,

$$G_{\mu_a} = \{g \in G | \mu_a(g) = \mu(a)\} = \{e, a^2, a^3\}$$

which is not a subgroup of G.

The following corollary is an immediate consequence of Proposition 3.2.4:

Corollary 3.3.1. $G_{\mu_x} = A^t_{\mu_x} = A^t_\mu x$; *for every* $x \in G$.

The following proposition gives a relation between G_μ and G_{μ_x}:

Proposition 3.3.1. *G_μ is a subset of G_{μ_x} for every $x \in G$.*

Proof. Let $g \in G_\mu \Rightarrow \mu(g) = \mu(e) \Rightarrow \mu(gx^{-1}) = \mu(x^{-1})$ (Proposition 1.4.4) $\Rightarrow \mu_x(g) = \mu(x) \Rightarrow g \in G_{\mu_x}$, and the result follows.

Now, we get a necessary and sufficient condition under which G_{μ_x} is a subgroup of G. The following lemma is needed.

Lemma 3.3.3. *If the product of any three elements in A^t_μ is again in A^t_μ, then $A^t_\mu x$ is a subgroup of G for every $x \in A^t_\mu$.*

Proof. Since μ is a fuzzy subgroup, therefore $x \in A^t_\mu$ implies that $x^{-1} \in A^t_\mu$ and hence $e = xx^{-1} \in A^t_\mu x$. Now, let g_1 and g_2 be any two elements of $A^t_\mu x$, then $g_1 = hx$ and $g_2 = kx$ for some $h, k \in A^t_\mu$. Thus $g_1 g_2 = (hx)(kx) = (hxk)x$. But $hxk \in A^t_\mu$ from the assumption. Thus $g_1 g_2 \in A^t_\mu x$. This completes the proof.

Proposition 3.3.2. *G_{μ_x} is a subgroup of G if and only if A_μ^t contains the product hxk for every $h,\ k \in A_\mu^t$.*

Proof. ($\Rightarrow$) Let $G_{\mu_x} = A_\mu^t x$ be a subgroup of G and let $g_1 = h_1 x,\ \ g_2 = h_2 x$ be any two elements of G_{μ_x}. Thus $g_1 g_2 = h_1 x h_2 x \in G_{\mu_x} \Rightarrow (h_1 x h_2) x \in A_\mu^t x \Rightarrow (h_1 x h_2) \in A_\mu^t$.

($\Leftarrow$) Let A_μ^t contain the product hxk for every $h, k \in A_\mu^t$ and let g_1 and g_2 be as above. Thus $g_1 g_2 = h_1 x h_2 x$. But $h_1 x h_2 \in A_\mu^t$, then $g_1 g_2 \in G_{\mu_x}$ and since $e \in G_{\mu_x}$, therefore G_{μ_x} is a subgroup of G.

The above results remain valid if μ_x is replaced by ${}_x\mu$.

Chapter Four

Fuzzy Quotient Groups and Quotient of Fuzzy Subgroups

In this chapter, we investigate further the concepts of fuzzy groups and quotient of fuzzy subgroups.

4.1 Fuzzy Quotient Groups

Mukherjee and Bhattacharya [19], [20] were the first to introduce the notion of fuzzy quotient groups in order to obtain a fuzzy analog to the famous Lagrange theorem. [19. Theorem 4.10]. More investigation on fuzzy quotient groups was done in [3], [11] and [14]. Our objective in this section is to establish some commutative diagrams that relate a fuzzy quotient group to some other known groups.

The following lemmas are needed before we recall the definition of a fuzzy quotient group. Although the results in these lemmas are known in [19], yet several steps in the proofs are different.

Lemma 4.1.1. *Let μ be a fuzzy normal subgroup of a group G. Then the composition $\mu_a \circ \mu_b = \mu_{ab}$ defined on the set of all fuzzy cosets of μ is well defined. [Recall* *that ${}_x\mu = \mu_x$ for all $x \in G$ since μ is fuzzy normal.]*

Proof. Let $x, y, a, b \in G$ such that

$$\mu_x = \mu_a \quad \text{and} \quad \mu_y = \mu_b$$

i.e., $x^{-1}a$ and $y^{-1}b \in G_\mu$ (Proposition 1.1.1). All we need to show is $\mu_x \circ \mu_y = \mu_a \circ \mu_b$, i.e., $\mu_{xy} = \mu_{ab}$ or, equivalently, $(xy)^{-1}(ab) \in G_\mu$. Since G_μ is normal in G (Proposition 2.1.1) and $x^{-1}a, y^{-1}b \in G_\mu$, therefore

$$(xy)^{-1}(ab) = y^{-1}x^{-1}ab = y^{-1}\left(x^{-1}a\right)y\left(y^{-1}b\right) \in G_\mu.$$

Hence, $\mu_{xy} = \mu_{ab}$, and the result follows.

The following corollary of Lemma 4.1.1 is straightforward:

Corollary 4.1.1. *The set $G/\mu = \{\mu_a | a \in G\}$ is a group under the composition defined in Lemma 4.1.1.*

Lemma 4.1.2. *Let $\overline{\mu}$ be a relation defined on G/μ by $\overline{\mu}(\mu_a) = \mu(a)$ for all $a \in G$. Then*

(i) $\overline{\mu}$ is a fuzzy subset of G/μ.

(ii) $\overline{\mu}$ is a fuzzy normal subgroup of G/μ.

Proof. (i) All we need to show is that $\overline{\mu}$ is well defined. Let $\mu_a = \mu_b \Rightarrow a^{-1}b \in G_\mu$ (Proposition 3.2.2) $\mu(a^{-1}b) = \mu(e) \Rightarrow \mu(a) = \mu(b)$ (Proposition 1.4.5) $\Rightarrow$

$\overline{\mu}(\mu_a) = \overline{\mu}(\mu_b) \Rightarrow \overline{\mu}$ is well defined and hence $\overline{\mu} : G/\mu \Rightarrow I$ is a fuzzy subset of G/μ.

(ii) Let $\mu_a, \mu_b \in G/\mu$, then

$$\begin{aligned}
\overline{\mu}(\mu_a \circ \mu_b) &= \overline{\mu}(\mu_{ab}) = \mu(ab) \geq \min(\mu(a), \mu(b)) \\
&= \min(\overline{\mu}(\mu_a), \overline{\mu}(\mu_a)).
\end{aligned}$$

Also, $\overline{\mu}(\mu_{a^{-1}}) = \mu(a^{-1}) = \mu(a) = \overline{\mu}(\mu_a)$. Thus, $\overline{\mu}$ is a fuzzy subgroup of G/μ. And since μ is fuzzy normal, therefore

$$\begin{aligned}
\overline{\mu}(\mu_a \circ \mu_b) &= \overline{\mu}(\mu_{ab}) = \mu(ab) = \mu(ba) = \overline{\mu}(\mu_{ba}) \\
&= \overline{\mu}(\mu_b \circ \mu_a) \Rightarrow \overline{\mu}
\end{aligned}$$

is a fuzzy normal subgroup of G/μ and the result follows.

Definition 4.1.1 [19]. The fuzzy subgroup $\overline{\mu}$ of G/μ defined in Lemma 4.1.2 is called the *fuzzy quotient group* determined by μ.

Lemma 4.1.3. *G/μ is a homomorphic image of G.*

Proof. Define the map $\phi : G \to G/\mu$ by $\phi(a) = \mu_a$ for all $a \in G$. Clearly, ϕ is well defined and onto. Let $a, b \in G$, then

$$\begin{aligned}
\phi(ab) &= \mu_{ab} = \mu_a \circ \mu_b \\
&= \phi(a) \circ \phi(b).
\end{aligned}$$

Thus ϕ is a homomorphism, and the result follows.

Now, let $G, \mu, G/\mu, \overline{\mu}$ and ϕ be as in the above lemmas. Then we immediately have

Proposition 4.1.2. *The following statements are equivalent:*

(i) $\overline{\mu}$ *is the fuzzy quotient group determined by* μ.

(ii) The following diagram commutes:

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\mu"'] & & G/\mu \arrow[dl, "\overline{\mu}"] \\ & I & \end{tikzcd}$$

i.e., $\mu(a) = \overline{\mu}(\phi(a))$ *for every* $a \in G$.

Proposition 4.1.3. *Let* $\phi : G \to G/\mu$ *be the homomorphism defined by* $\phi(a) = \mu_a$ *for all* $a \in G$. *Then*

(i) $\mathrm{Ker}(\phi) = G_\mu$.

(ii) The groups G/G_μ *and* G/μ *are isomorphic.*

Proof. (*i*)
$$\begin{aligned} \mathrm{Ker}(\phi) &= \{a \in G \mid \phi(a) = \mu_e = \mu\} \\ &= \{a \in G \mid \mu_a = \mu_e\} \\ &= \{a \in G \mid a \in G_\mu\} \quad \text{(Proposition 3.2.1)} \\ &= G_\mu. \end{aligned}$$

(ii) Since $\mathrm{Ker}(\phi) = G_\mu$, therefore the first isomorphism theorem of group gives immediately

$$G/G_\mu \cong G/\mu,$$

i.e., there exists an isomorphism $\psi : G/G_\mu \to G/\mu$ which completes the proof.

The following theorem combines the results obtained in this section and describes the relation between the fuzzy quotient group $\overline{\mu}$ and the other groups by means of a commutative diagram:

Theorem 4.1.1. *Let $\phi_\mu : G \to G/G_\mu$ be the canonical homomorphism. Then the following diagram commutes:*

$$\begin{tikzcd}
& G/G_\mu \arrow[dr, "\psi"] & \\
G \arrow[ur, "\phi_\mu"] \arrow[rr, "\phi"] \arrow[dr, "\mu"'] & & G/\mu \arrow[dl, "\overline{\mu}"] \\
& I &
\end{tikzcd}$$

Now we give a counter-example to show the statement "If μ and ν are fuzzy normal subgroups of a group G such that $\mu \subseteq \nu$, then the groups $(G/\mu)/(G_\nu/\mu)$ and G/ν are isomorphic". which is given as Proposition 2.3.2 in [18] is incorrect as shown in the following counter-example:

Example 4.1.1. Consider the following fuzzy normal subgroups μ and ν of the Klein-four group $V = \{e, a, b, ab\}$ given by

$$\mu : \mu_{t_0} = G_\mu \subsetneq \mu_{t_1} = G = V$$

$$\nu : \nu_{t_2} = G_\nu \subsetneq \nu_{t_3} = G = V,$$

where

$$A_\mu^{t_0} = \{a, e\}, \quad A_\mu^{t_1} = \{b, ab\}$$

$$A_\nu^{t_2} = \{e, b\}, \quad A_\nu^{t_3} = \{a, ab\}$$

and

$$t_0 < t_1 < t_2 < t_3.$$

Thus,

$$\begin{aligned} G/\mu &= \{\mu_e = \mu_a, \quad \mu_b = \mu_{ab}\} \\ G/\nu &= \{\mu_e = \mu_b, \quad \mu_a = \mu_{ab}\} \\ G_\nu/\mu &= \{\mu_e, \mu_b\} \end{aligned}$$

$(G/\mu)/(G_{\nu/\mu})$ is order 1. while $G_{\nu/\mu}$ is order 2 and hence cannot be isomorphic.

We now give a correct version of the above stated Proposition 2.3.2 in [18]. The following lemma is needed.

Lemma 4.1.3. *Let μ and ν be two fuzzy normal subgroups of a group G such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. Then G_μ is a subgroup of G_ν.*

Proof. All we need to show is that G_μ is a subset of G_ν. Let $x \in G_\mu \Rightarrow \mu(x) = \mu(e) = \nu(e)$. But $\mu(x) \leq \nu(x)$ for all $x \in G$. Thus $\mu(x) = \nu(e) \leq \nu(x)$ and hence $\nu(x) = \nu(e)$, i.e., $x \in G_\nu$ and the result follows.

Proposition 4.1.2. *Let μ and ν be as in lemma 4.1.3. Then the groups $(G/\mu)/(G_\nu/\mu)$ and G/ν isomorphic.*

Proof. Define the map $\phi : G/\mu \rightarrow G/\nu$ as $\phi(\mu_a) = \nu_a$ for all $a, b \in G$. The mapping ϕ is well defined since $\mu_a = \mu_b \Rightarrow ab^{-1} \in G_\mu \Rightarrow \nu(e) = \mu(e) = \mu(ab^{-1}) \leq \nu(a^{-1}(b)) \Rightarrow \nu(a^{-1}b) = \nu(e) \Rightarrow a^{-1}b \in G_\nu \Rightarrow \nu_a = \nu_b$ and hence ϕ is well-defined. ϕ is an onto-homomorphism since

$$\begin{aligned} \phi(\mu_a \circ \mu_b) &= \phi(\mu_{ab}) = \nu_{ab} = \nu_a \circ \nu_b \\ &= \phi(\mu_a) \circ \phi(\mu_b). \end{aligned}$$

The kernel of ϕ is given by

$$\begin{aligned} \mathrm{Ker}(\phi) &= \{\mu_a | \phi(\mu_a) = \nu_e\} \\ &= \{\mu_a | \nu_a = \nu_e\} \\ &= \{\mu_a | a \in G_\nu\} \\ &= G_\nu/\mu. \end{aligned}$$

Thus, by the first isomorphism theorem of groups, we have $(G/\mu)/(G_\nu/\mu)$ isomorphic to G/ν, and the result follows.

4.2 Quotients of Fuzzy Subgroups and Fuzzy Solvable Groups

In [7] Bhattacharya and Mukherjee introduced the notion of quotient of fuzzy groups of the form μ/ν for some fuzzy normal subgroups μ and ν of a group G. Then they constructed certain chains of fuzzy normal subgroups and fuzzy quotient groups of the type μ/ν in order to define the notion of fuzzy solvable groups. The main objective of this section is to straighten out these definitions of fuzzy solvable groups which apparently need some modifications due to their wrong construction of the chain of quotients of fuzzy subgroups. But first we need the following lemma:

Lemma 4.2.1. *Let μ and ν be two fuzzy subgroups of a group G such that μ is fuzzy normal, $\mu \leq \nu$ and $\mu(e) = \nu(e)$. The map $\nu/\mu : G/G_\mu \to I$ defined by $(\nu/\mu)(aG_\mu) = \nu(a)$ is a fuzzy subgroup of G.*

Proof. We need to show first that the map μ/ν is well defined. Let $aG_\mu = bG_\mu \Rightarrow a^{-1}b \in G_\mu \Rightarrow \mu(a^{-1}b) = \mu(e) \Rightarrow \mu(a^{-1}b) = \nu(e) \leq \nu(a^{-1}b)$ (from the hypothesis) $\Rightarrow \nu(a^{-1}b) = \nu(e) \Rightarrow \nu(a) = \nu(b)$, and hence the map μ/ν is well defined.

Now $(\nu/\mu)(aG_\mu bG_\mu) = (\nu/\mu)(abG_\mu) = \nu(ab) \geq \min(\nu(a), \nu(b)) = \min((\nu/\mu)(aG_\mu), (\nu/\mu)(bG_\mu))$ and $(\nu/\mu)(a^{-1}G_\mu) = \nu(a^{-1}) = \nu(a) = (\nu/\mu)(aG_\mu)$ which completes the proof.

Definition 4.2.1. With the hypothesis of Lemma 4.1, the fuzzy subgroup ν/μ of

G/G_μ is called the quotient of ν by μ.

The fuzzy subgroup ν in the above definition is not required to be fuzzy normal as in the definition of quotient of fuzzy subgroups given in [7]

Corollary 4.2.1 [7. Lemma 5.1]. *If μ and ν are as in Lemma 4.1 with the additional condition that ν is fuzzy normal, then ν/μ is a fuzzy normal subgroup of G/G_μ.*

Proof. Straightforward.

The following lemma is an immediate consequence of Lemma 4.1.3:

Lemma 4.2.2 [7]. *Let $\nu_1 \subseteq \nu_2 \subseteq \cdots \subseteq \nu_k$ be a chain of fuzzy normal subgroups of a group G, $\nu_i(e) = \nu_1(e)$ for all $1 \leq i \leq k$. Then $G_{\nu_1} \subseteq G_{\nu_2} \subseteq \cdots \subseteq G_{\nu_k}$.*

Proof. Similar to the proof of Lemma 4.1.3.

Bhattacharya and Mukherjee [7] claim that the chain $\nu_1 \subseteq \nu_2 \subseteq \cdots \subseteq \nu_k$ in lemma 4.2.2 yields another chain of quotients of fuzzy subgroups

$$\eta_1 \subseteq \eta_2 \subseteq \cdots \subseteq \eta_k$$

where

$$\eta_i = \nu_{i+1}/\nu_i \qquad 1 \leq i \leq k-1.$$

This claim is clearly wrong since the η's are of different domains and hence the chain $\eta_1 \subseteq \eta_2 \subseteq \cdots \subseteq \eta_k$ is meaningless. Consequently, their definition [7, p. 89] of fuzzy solvable group is incorrect. We state their definition for completion.

"A fuzzy normal subgroup μ of a group G is fuzzy solvable if there exists a chain of fuzzy normal subgroups

$$\nu_1 \subseteq \nu_2 \subseteq \cdots \subseteq \nu_k = \mu$$

with $\nu_1(x) = \nu_1(e)$ only when $x = e$ and $\nu_i(e) = \nu_1(e), \quad 1 \le i \le k$ such that there is a corresponding chain of fuzzy abelian subgroups $\eta_1 \subseteq \eta_2 \subseteq \cdots \subseteq \eta_k$ where $\eta_i = \nu_{i+1}/\nu_i, \quad 1 \le i \le k$."

Now we give a correct version of the definition of fuzzy solvable groups.

Definition 1.4.2. A fuzzy normal subgroup μ of a group G is fuzzy solvable if there exists a chain of fuzzy normal subgroups

$$\nu_1 \subseteq \nu_2 \subseteq \cdots \subseteq \nu_k = \mu$$

with $\nu_1(x) = \nu_1(e)$ only when $x = e$ and $\nu_i(e) = \nu_1(e), \quad 1 \le i \le k$ such that quotient of fuzzy subgroups $\nu_{i+1}/\nu_i, \quad 1 \le i \le k-1$ is fuzzy abelian.

Example 4.2.1. Let G be the dihedral group

$$G = D_4 = \langle a, b | a^4 = b^2 = e, a^i b = ba^{-1} \rangle$$

and let ν_1, ν_2 and ν_3 be the fuzzy normal subgroups of G where the annuli can be tabulated as follows:

ν_1 :

t_0-annulus	t_1-annulus	t_2-annulus	t_3-annulus
e	a^2	b, a^2b	a, a^3, ab, a^3b

ν_2 :	t_0-annulus	t_4-annulus	t_5-annulus
	e, a^2	b, a^2b	a, a^3, ab, a^3b

ν_3 :	t_0-annulus	t_6-annulus
	e, a^2, b, a^2b	a, a^3, ab, a^3b

where

$$t_0 > t_1 > t_2 > t_3, t_4 > t_2$$

and

$$t_6 > t_5 > t_3.$$

Therefore,

$$\nu_1 \subseteq \nu_2 \subseteq \nu_3.$$

Now, the annuli of the quotients of fuzzy subgroups:

$$\nu_2/\nu_1 : G/G_{\nu_1} \to I,$$

and

$$\nu_3/\nu_2 : G/G_{\nu_1} \to I$$

can be tabulated as follows:

ν_2/ν_1 :	t_0-annulus	t_4-annulus	t_5-annulus
	ϵ, a^2	b, a^2b	a, a^3, ab, a^3b

ν_3/ν_2 :	t_0-annulus	t_6-annulus
	G_{ν_1}, bG_{ν_1}	aG_{ν_1}, abG_{ν}

Since the factor groups $((G/G_{\nu_1})/(G/G_{\nu_1}))_{\nu_2/\nu_1}$ and $((G/G_{\nu_2})/(G/G_{\nu_2}))_{\nu_3/\nu_2}$ are abelian, therefore ν_2/ν_1 and ν_3/ν_2 are fuzzy abelian (Proposition 2.3.2). Hence, ν_3 is a fuzzy solvable group.

The fact that any subgroup of a solvable group is solvable is no longer true in the fuzzy context as shown in the following counter-example.

Example 4.2.2. Consider the fuzzy normal subgroups ν_1, ν_2 and ν_3 of D_4 given in Example 4.2.1. We have $\nu_1 \subseteq \nu_2$ with ν_2/ν_1 is fuzzy abelian and hence ν_2 is fuzzy solvable, whereas it is clear that ν_1 is not fuzzy solvable.

The above discussion yields the following proposition whose proof is now straightforward.

Proposition 4.2.1. *A group G is solvable if and only if G has a fuzzy solvable subgroup.*

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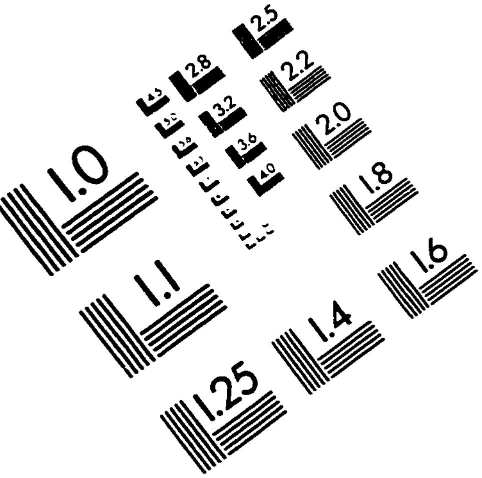
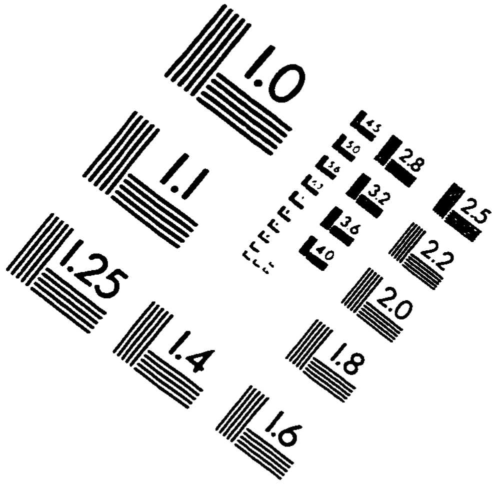
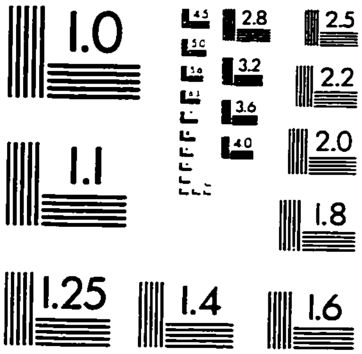
IMAGE EVALUATION TEST TARGET (QA-3)

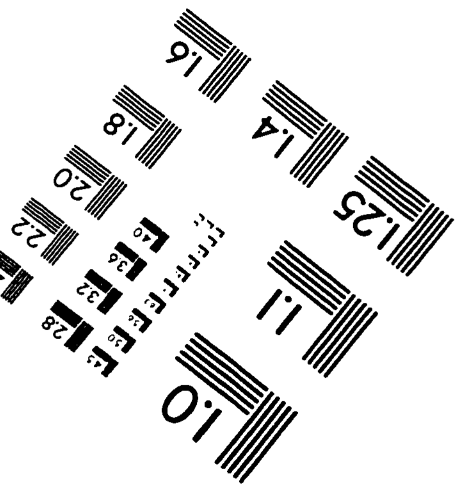


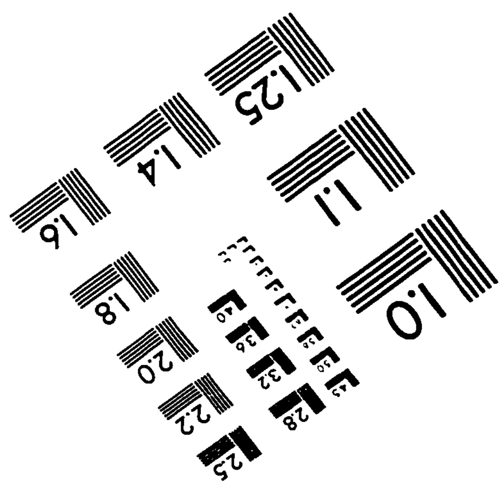




150mm

6″





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